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Chiral Solitons in $1+2$ Dimensions

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Abstract This paper obtains the exact 1-soliton solution to the chiral nonlinear Schrödinger's equation in $1+2$ dimensions. Both constant coefficients and time-dependent coefficients are considered. The topological as well as bright soliton solutions are obtained. The soliton ansatz method is used to carry out the derivation of the soliton.

Keywords Chiral solitons · Integrability

1 Introduction

Chiral solitons are studied in the context of Quantum Hall effect where chiral excitations are known to appear. The model in this context is known as the Jackiw and Pi model. This model is given by the chiral nonlinear Schrödinger's equation (NLSE) [1–10]. In this paper, the chiral NLSE will be studied in $1+2$ dimensions with both constant as well as time-dependent coefficients.

The chiral NLSE in $1+1$ dimensions was solved for topological as well as nontopological solitons in 1998 by Nishino et al. [9]. Later, this problem was studied with Bohm potential in 2004 and 2009 [2, 8]. This study was then extended to the case of chiral solitons in $1+1$ dimensions with time-dependent coefficients [1]. In this paper, the study is going to be carried out for chiral NLSE in $1+2$ dimensions, both with constant as well as with time-dependent coefficients. It is important to consider the time-dependent coefficients to the chiral NLSE as this is more close to reality. It will be seen that for the case of time-dependent coefficients, the only criterion for the chiral solitons to exist is that the dispersion coefficient must be Riemann integrable.

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2 Constant Coefficients

The chiral NLSE in $1+2$ dimensions, with constant coefficients, that is going to be studied in this paper is given by

$$iq_t + a(q_{xx} + q_{yy}) + i\{b_1(qq_x^* - q^*q_x) + b_2(qq_y^* - q^*q_y)\}q = 0 \tag{1}$$

In (1), the first term is the evolution term, while a represents the coefficient of dispersion term. Also, b_1 and b_2 are the coefficients of nonlinear coupling terms. This nonlinearity is known as the current density. It needs to be noted that (1) is not Galilean invariant and also is not integrable by the method of Inverse Scattering Transform, since it will fail the Painleve test of integrability.

In order to solve (1), it is assumed that the soliton solution to (1), is given in the form [9]:

$$q(x, y, t) = P(x, y, t)e^{i\phi(x,y,t)} \tag{2}$$

where $P(x, y, t)$ is the amplitude portion of the soliton, while the phase portion of the soliton is given by

$$\phi(x, y, t) = \kappa_1 x + \kappa_2 y + \omega t + \theta \tag{3}$$

Here in (3), κ_1 and κ_2 are the frequencies in the x- and y-directions, ω is the soliton frequency while θ is the phase constant. Thus from (2) and (3),

$$iq_t = \left(i\frac{\partial P}{\partial t} - \omega P\right)e^{i\phi} \tag{4}$$

$$q_{xx} = \left(\frac{\partial^2 P}{\partial x^2} + 2i\kappa_1\frac{\partial P}{\partial x} - \kappa_1^2 P\right)e^{i\phi} \tag{5}$$

$$q_{yy} = \left(\frac{\partial^2 P}{\partial y^2} + 2i\kappa_2\frac{\partial P}{\partial y} - \kappa_2^2 P\right)e^{i\phi} \tag{6}$$

$$(qq_x^* - q^*q_x)q = -2i\kappa_1 P^3 e^{i\phi} \tag{7}$$

$$(qq_y^* - q^*q_y)q = -2i\kappa_2 P^3 e^{i\phi} \tag{8}$$

Substituting (4)–(8) into (1) and decomposing into real and imaginary parts yields, respectively

$$\omega P - a(t)\left\{\left(\frac{\partial^2 P}{\partial x^2} - \kappa_1^2 P\right) + \left(\frac{\partial^2 P}{\partial y^2} - \kappa_2^2 P\right)\right\} + 2P^3(\kappa_1 b_1 + \kappa_2 b_2) = 0 \tag{9}$$

$$\frac{\partial P}{\partial t} + 2a\left(\kappa_1\frac{\partial P}{\partial x} + \kappa_2\frac{\partial P}{\partial y}\right) = 0 \tag{10}$$

This pair of equations will be analyzed further depending on the type of soliton solution is being fetched. The analysis is now carried out in the following two subsections.

2.1 Bright Solitons

Bright solitons are also known as bell-shaped solitons or nontopological solitons. These kind of solitons are modeled by the sech function. Therefore, the hypothesis for the function

$P(x, y, t)$ will be given by [2, 9]

$$P(x, y, t) = \frac{A}{\cosh^p \tau} \tag{11}$$

where

$$\tau = B_1 x + B_2 y - vt \tag{12}$$

Here in (11) and (12) A is the soliton amplitude, while B_1 and B_2 are the inverse width of the soliton in x- and y-directions respectively and v is the soliton velocity. The index p is unknown at this point and its value will be derived during the course of the derivation of the solution to (1). The parameters A, B_1 and B_2 are all constants. Thus, from the ansatz given by (11), the equation pair (9) and (10) reduces to

$$\begin{aligned} &\frac{\omega A}{\cosh^p \tau} - \frac{2A^3(\kappa_1 b_1 + \kappa_2 b_2)}{\cosh^{3p} \tau} \\ &\quad - aA\left\{\frac{p^2(B_1^2 + B_2^2) - (\kappa_1^2 + \kappa_2^2)}{\cosh^p \tau} - \frac{p(p+1)(B_1^2 + B_2^2)}{\cosh^{p+2} \tau}\right\} = 0 \end{aligned} \tag{13}$$

and

$$pAv\frac{\tanh \tau}{\cosh^p \tau} - 2paA(\kappa_1 B_1 + \kappa_2 B_2)\frac{\tanh \tau}{\cosh^p \tau} = 0 \tag{14}$$

From (13) setting the exponents $3p$ and $p+2$ equal to one another yields

$$3p = p + 2 \tag{15}$$

which gives

$$p = 1 \tag{16}$$

Now from (13) noting that $1/\cosh^{p+j} \tau$ are linearly independent functions for $j = 0, 2$, its coefficients must be respectively set to zero. This leads to

$$\omega = a(B_1^2 + B_2^2 - \kappa_1^2 - \kappa_2^2) \tag{17}$$

and

$$A = \sqrt{\frac{a(B_1^2 + B_2^2)}{\kappa_1 b_1 + \kappa_2 b_2}} \tag{18}$$

which shows that it is necessary to have

$$a(\kappa_1 b_1 + \kappa_2 b_2) > 0 \tag{19}$$

and that

$$\frac{a}{\kappa_1 b_1 + \kappa_2 b_2} = \text{constant} \tag{20}$$

since A, B_1 and B_2 are constants. Now, from (14), setting the coefficients of the linearly independent function $\tanh\tau/\cosh^p\tau$ to zero yields

$$v = 2a(\kappa_1 B_1 + \kappa_2 B_2) \tag{21}$$

Thus, the bright chiral solitons are given by

$$q(x,y,t) = \frac{A}{\cosh(B_1 x + B_2 y - vt)} e^{i(\kappa_1 x + \kappa_2 y + \omega t + \theta)} \tag{22}$$

where the relation between the solitons amplitude A and the widths B_1 and B_2 is given by (18), while the soliton wave number and velocity are respectively seen in (17) and (21).

2.2 Dark Solitons

The dark solitons are also known as topological solitons or simply topological defects. The hypothesis in this case is given by

$$P(x,y,t) = A\tanh^p\tau \tag{23}$$

where τ is given in (12). For topological solitons, the parameters A, B_1 and B_2 are free parameters. Thus, the real and imaginary parts given by (9) and (10) respectively reduce to

$$\begin{aligned} &\omega A\tanh^p\tau + 2A^3(\kappa_1 b_1 + \kappa_2 b_2)\tanh^{3p}\tau \\ &\quad + aA[p(B_1^2 + B_2^2)\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\} \\ &\quad - (\kappa_1^2 + \kappa_2^2)\tanh^p\tau] = 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} &pAv(\tanh^{p-1}\tau - \tanh^{p+1}\tau) \\ &\quad - 2paA(\kappa_1 B_1 + \kappa_2 B_2)(\tanh^{p-1}\tau - \tanh^{p+1}\tau) = 0 \end{aligned} \tag{25}$$

From (24), setting the exponents $3p$ and $p+2$ leads to the same value of p as in (16). Now, setting the coefficients of the linearly independent functions $\tanh^{p+j}$ to zero for $j = 0, 2$ leads to

$$\omega = a\{2(B_1^2 + B_2^2) - (\kappa_1^2 + \kappa_2^2)\} \tag{26}$$

and

$$A = \sqrt{-\frac{a(B_1^2 + B_2^2)}{\kappa_1 b_1 + \kappa_2 b_2}} \tag{27}$$

which shows that it is necessary to have

$$a(\kappa_1 b_1 + \kappa_2 b_2) < 0 \tag{28}$$

It needs to be noted that the coefficients of the third linearly independent function $\tanh^{p-2}\tau$ are automatically zero due to the value of p given in (16). Again from the imaginary part given by (25), the same value of the velocity of the soliton, as in (21), is obtained. Thus, finally the topological chiral soliton is given by

$$q(x, y, t) = A \tanh(B_1 x + B_2 y - vt) e^{i(\kappa_1 x + \kappa_2 y + \omega t + \theta)} \tag{29}$$

where the relation between the free parameters A, B_1 and B_2 are given in (27) while the wave number is given by (26). These induce restrictions on the soliton parameters and coefficients that are given by (20) and (28), for topological solitons to exist.

3 Time-Dependent Coefficients

The chiral NLSE with time-dependent coefficients is given by

$$iq_t + a(t)(q_{xx} + q_{yy}) + i\{b_1(t)(qq_x^* - q^* q_x) + b_2(t)(qq_y^* - q^* q_y)\}q = 0 \tag{30}$$

where the coefficients in (30) are now all time-dependent. In order to solve (30), the same assumption as in (2) is carried out. Now, it needs to be noted that because of time-dependent coefficients, the frequencies and wave number are all time-dependent. Thus, (3), in this case modifies to

$$\phi(x, y, t) = \kappa_1(t)x + \kappa_2(t)y + \omega(t)t + \theta(t) \tag{31}$$

Therefore,

$$iq_t = \left\{ i\frac{\partial P}{\partial t} - P\left(x\frac{d\kappa_1}{dt} + y\frac{d\kappa_2}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt} \right) \right\} e^{i\phi} \tag{32}$$

while (5)–(8) stays the same. Now, substituting (32) and (5)–(8) in (30) and decomposing into real and imaginary parts yields

$$\begin{aligned} &P\left(x\frac{d\kappa_1}{dt} + y\frac{d\kappa_2}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt} \right) \\ &\quad - a(t)\left\{ \left(\frac{\partial^2 P}{\partial x^2} - \kappa_1^2 P \right) + \left(\frac{\partial^2 P}{\partial y^2} - \kappa_2^2 P \right) \right\} + 2P^3(\kappa_1 b_1 + \kappa_2 b_2) = 0 \end{aligned} \tag{33}$$

while the imaginary part stays the same as in (10). These equations will now be analyzed and solved to obtain the exact 1-soliton solution to (30). The study will now be divided into the following two subsections.

3.1 Bright Solitons

For bright solitons, the ansatz stays the same as in (11). The difference is that the amplitude A and the widths B_1 and B_2, given in (11) and (12) are now functions of time. Thus, the real and imaginary parts given by (33) and (10) respectively change to

$$\begin{aligned} &\frac{A}{\cosh^p \tau}\left(x\frac{d\kappa_1}{dt} + y\frac{d\kappa_2}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt} \right) \\ &\quad - a(t)A\left\{ \frac{p^2(B_1^2 + B_2^2) - (\kappa_1^2 + \kappa_2^2)}{\cosh^p \tau} - \frac{p(p+1)(B_1^2 + B_2^2)}{\cosh^{p+2} \tau} \right\} = 0 \end{aligned} \tag{34}$$

and

$$\frac{dA}{dt}\frac{1}{\cosh^p\tau} - pA\frac{\tanh\tau}{\cosh^p\tau}\left\{x\frac{dB_1}{dt} + y\frac{dB_2}{dt} - \left(v + t\frac{dv}{dt}\right)\right\}$$
$$- 2pa(t)A(\kappa_1 B_1 + \kappa_2 B_2)\frac{\tanh\tau}{\cosh^p\tau} = 0 \tag{35}$$

By the same process as in the constant coefficients case, the wave number and the amplitude-width relationship stays the same as in (17) and (18). The constraint relations also stay the same. Also, from (34) it is possible to conclude that the frequencies κ_1, κ_2, the wave number ω and the phase constant θ all stay constant.

Finally, from (35), setting the coefficient of the linearly independent function $1/\cosh^p\tau$ to zero gives

$$\frac{dA}{dt} = 0 \tag{36}$$

which shows that the amplitude A is a constant. Similarly, the widths B_1 and B_2 are also constants. Now setting the coefficients of $\tanh\tau/\cosh^p\tau$ gives

$$\frac{d}{dt}(vt) = 2a(t)(\kappa_1 B_1 + \kappa_2 B_2) \tag{37}$$

which leads to

$$v(t) = \frac{2(\kappa_1 B_1 + \kappa_2 B_2)}{t}\int a(t)dt \tag{38}$$

Thus, the velocity of the soliton is not a constant in the case of time-dependent coefficient. This velocity will be meaningful provided the function $a(t)$ is Riemann integrable.

3.2 Dark Solitons

For dark solitons with time-dependent coefficients, the starting point is the same hypothesis as given in (23). Here, as in the case of bright solitons with time-dependent coefficients, the soliton parameters are all time-dependent. Thus, substituting the hypothesis into (33) and (10) respectively yields

$$A\tanh^p\tau\left(x\frac{d\kappa_1}{dt} + y\frac{d\kappa_2}{dt} + \omega + t\frac{d\omega}{dt} + \frac{d\theta}{dt}\right)$$
$$+ 2A^3(\kappa_1 b_1 + \kappa_2 b_2)\tanh^{3p}\tau$$
$$+ aA[p(B_1^2 + B_2^2)\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\}$$
$$- (\kappa_1^2 + \kappa_2^2)\tanh^p\tau] = 0 \tag{39}$$

and

$$\frac{dA}{dt}\tanh^p\tau + 2a(t)pA(\kappa_1 B_1 + \kappa_2 B_2)(\tanh^{p-1}\tau - \tanh^{p+1}\tau)$$
$$+ pA(\tanh^{p-1}\tau - \tanh^{p+1}\tau)\left\{x\frac{dB_1}{dt} + y\frac{dB_2}{dt} - \left(v + t\frac{dv}{dt}\right)\right\} = 0 \tag{40}$$

Proceeding as before, the same value of p is yielded. Also, the relation between the free parameters is the same as in the case of constant coefficients. From the imaginary part relation (40), is again true that the free parameters A, B_1 and B_2 are all constants and the velocity $v(t)$ is given as in (38). The same domain restrictions are valid as in constant coefficient case.

4 Conclusions

This paper obtains the topological and bright soliton solution to the chiral NLSE in $1 + 2$ dimensions. Both the cases were considered, namely with constant coefficients and time-dependent coefficients. This analysis was carried out with respect to both topological as well as nontopological solitons.

In future, this study will be extended to the case with perturbation terms. The soliton perturbation theory will be established and the adiabatic parameter dynamics of these solitons will be studied. Better yet, the quasi-stationary solitons as well as the quasi-particle theory of chiral solitons will also be developed. Those results will be reported in the future.

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